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In the woods of M-theory

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Abstract: We study BPS states which arise in compactifications of M-theory on Calabi-Yau manifolds. In particular, we are interested in the spectrum of the particles obtained by wrapping $M2$-brane on a two-cycle in the CY manifold X. We compute the Euler characteristics of the moduli space of genus zero curves which land in a holomorphic four-cycle $S \subset X$. We use M. Kontsevich's method which reduces the problem to summing over trees and observe the discrepancy with the predictions of local mirror symmetry. We then turn this discrepancy into a supporting evidence in favor of existence of extra moduli of $M2$-branes which consists of the choice of a flat U(1) connection recently suggested by C. Vafa and partially confirm this by counting of the arbitrary genus curves of bi-degree $(2, n)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ (this part has been done together with Barak Kol). We also make a conjecture concerning the counting of higher genus curves using second quantized Penner model and discuss possible applications to the string theory of two-dimensional QCD.

Keywords: M-Theory, Sigma Models, Nonperturbative Effects.

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1. Introduction

The study of M-theory [1] is a fascinating problem. One of the hints towards its existence is the hidden five dimensional structure in the geometry of four dimensional $\mathcal{N} = 2$ gauge theories. For example, if the gauge theory is realized as type-IIA string theory compactified on a (perhaps singular) Calabi-Yau manifold X then its vector multiplet structure is encoded in the prepotential:

$$\mathcal{F} = \sum_{\vec{n}} N_{\vec{n}} \mathrm{Li}_3(e^{-\vec{n}\cdot\vec{t}}), \tag{1.1}$$

where $\vec{t}$ are the special coordinates on the Kahler moduli space of X, $\vec{n}$ runs through a set of integral points in the positive Kahler cone and $N_{\vec{n}}$ is the “number” of the rational curves representing the homology class $\vec{n}$. The word “number” is in quote marks since it may be rational due to singularities of the moduli space of curves.

The promised five dimensional structure is in the fact that when translated to the effective coupling the expression (1.1) can be written as follows:

$$\tau_{ij} = \frac{\partial \mathcal{F}}{\partial t_i \partial t_j} = \sum_{\vec{n}} n_i n_j N_{\vec{n}} \sum_{k\in\mathbb{Z}} \log\left(\vec{t}\cdot\vec{n} + 2\pi k\right) + \log\Lambda\,. \tag{1.2}$$

In this representation we replaced $Li_1(e^{-a}) \equiv -\log(1-e^{-a})$ by

$$\frac{a}{2} - \log\left(2\sinh\frac{a}{2}\right) = \frac{a}{2} - \sum_{k\in\mathbb{Z}} \log\left(2\pi n + a\right) + \log\Lambda\,,$$

where Λ is a divergent constant. This representation is most naturally interpreted as a sum over all Kaluza-Klein charged states [2].

The paper is organized as follows. In the section 2 we remind the setup of local mirror symmetry and describe the non-compact Calabi-Yau manifolds which we are going to study. In the section 3 we proceed with counting of curves in genus zero. The section 4 (speculative at the moment) contains a comparison to the predictions of mirror symmetry and partial resoltuion of the contradiction which appears in the course of such comparison. The section 5 contains concluding remarks and speculations about counting of higher genus curves, applications to the string theory of QCD_2, and other remarks.

2. The setup

2.1 The local geometry

Consider the Calabi-Yau manifoild X which contains a surface S with $b_2^+ = 1$. The normal bundle to S is necessarily isomorphic to its canonical bundle K_S and the latter has no sections, since $2h^{2,0}(S) = b_2^+ - 1 = 0$. In computing the prepotential of the four dimensional theory obtained by compactification of IIA string on X one encounters a problem of counting the holomorphic curves in X which land in S. When lifted to M-theory the same question arises in the study of BPS particles obtained by wrapping the $M2$-brane around two-cycles β in S.

For the generic complex structure on X there are no holomorphic surfaces S embedded into X. Also, the holomorphic curves in X are generically isolated. What it means is that the problem of counting curves in X and the problem of counting curves in S are different. Nevertheless, it is clear that given the complex structure on X which is such that S is a holomorphic submanifold in X one encounters a *moduli* space $\overline{\mathcal{M}} = \amalg \overline{\mathcal{M}}_\beta$ of curves landing in S. It turns out that on top of this moduli space one also encounters the *obstruction bundle* $\mathcal{F} = \amalg \mathcal{F}_\beta$. The meaning of this bundle is roughly the following. Take a holomorphic curve in S and start varying the complex structure of X. The generic curve will cease to remain holomorphic. Only some special points in $\mathcal{M}_\beta$ correspond to the curves which remain holomorphic. These points can be identified with the zeroes of some section of $\mathcal{F}_\beta$. The number of zeroes counted with appropriate multiplicity is nothing but the Euler number of the bundle $\mathcal{F}_\beta$. Hence the prepotential (1.1) receives a contribution which is proportional to

$$N_\beta = \int_{\overline{\mathcal{M}}_\beta} \mathrm{Eu}\mathcal{F}_\beta\,. \tag{2.1}$$

It is important that $\mathrm{rk}\mathcal{F}_\beta = \dim\overline{\mathcal{M}}_\beta$ otherwise $N_\beta = 0$.

2.2 The strategy

We are going to study the toric S. The variety S is acted upon by the two dimensional torus T^2. The quotient S/T^2 is a convex polygon Δ due to a theorem by M. Atiyah and has the following meaning. Its vertices f correspond to the fixed points of the torus action while the edges connecting two vertices f_1, f_2 correspond to the degenerate orbits. The union of all degenerate orbits which sit on top of the given edge is isomorphic to a two-sphere. Let $\vec{d}_{f_1,f_2}$ be the homology class of this sphere. We denote by $t_{f_1,f_2} = \vec{t} \cdot \vec{d}_{f_1,f_2}$.

The torus $\mathbf{T} = \mathbb{C}^* \times \mathbb{C}^*$ which is a complexification of T^2 acts on S. It also acts on the moduli space of stable maps of the curves into S. The latter is a disjoint union of the spaces $\overline{\mathcal{M}}_{g,\beta}$ where g denotes the genus of a generic curve C and β the degree of the map: $\beta = [C] \in H_2(S)$.

The fixed points of the action of $\mathbf{T}$ on $\overline{\mathcal{M}}_{g,k,\beta}$ were described by M. Kontsevich in [3]. We first describe them in words and then proceed to enumerating them. Roughly speaking the only possibility for the stable curve to be fixed under the $\mathbf{T}$ action is to be a union of genus zero curves which are mapped to the spheres connecting the fixed points in S and the rest of the components mapped to the fixed points. In other words the normalization $\widetilde{C}$ of C splits as follows:

$$\widetilde{C} = \amalg_\alpha C_\alpha \,, \tag{2.2}$$

where for each C_α there are two options. If C_α has genus higher then zero or it has more then two special (marked or singular) points, then it is mapped to one of the fixed points $f \in S$. If C_α is of genus zero and has exactly two special points then it is mapped to the sphere connectiong f_1, f_2 where f_1 and f_2 are the fixed points in S which are at the same time the images of the two special points in C_α. The component of $\overline{\mathcal{M}}^{\mathbf{T}}_{g,\beta}$ which contains this curve is isomorphic to $\prod_v \overline{\mathcal{M}}_{g_v,n_v}$. Here the product is taken over all components which are mapped to points. The paper [3] explicitly describes all components of the fixed set. They are enumerated by the equivalence classes of the connected graphs Γ (E_Γ, V_Γ denote respectively the set of edges and the set of vertices of Γ) with the following data.

1. To each vertex $v \in V_\Gamma$ a vertex in Δ is assigned : $v \mapsto f_v$.

2. To each edge e an integer $d_e > 0$ is assigned.

3. If an edge connects two vertices v_1, v_2 then $f_{v_1} \neq f_{v_2}$.

4. $\sum_{e \in E_\Gamma} d_e \cdot \vec{d}_{f_{v_1},f_{v_2}} = \vec{d}$

5. To each vertex v a set $S_v \subset \{1, \ldots k\}$ of the marked points which belong to the component C_v of a curve is assigned. $\{1, \ldots, k\} = \amalg_{v \in V_\Gamma} S_v$

6. To each vertex a number g_v (an internal genus) is assigned. The stability implies that $2-2g_v-n_v<0$.

The total genus of the curve C is computed as

$$g = 1 + \#E_\Gamma + \sum_{v\in V_\Gamma}(g_v-1)\,. \qquad (2.3)$$

3. Counting in genus zero

3.1 Preliminary construction without marked points

If we restrict ourselves to the genus zero maps $g=0$ then the graphs are necessarily trees and our computation reduces to the sum over all trees of the certain weights. The vertex v of valency n_v contributes a factor $\chi(\overline{\mathcal{M}}_{0,n_v})$. Let us introduce a notation: $\chi(\overline{\mathcal{M}}_{0,1}) = \chi(\overline{\mathcal{M}}_{0,2}) = 1$. Then the sum of all contributions of the fixed components can be written as:

$$\sum_{\Gamma,n,\vec{d}} \frac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{v\in V_\Gamma} \chi(\overline{\mathcal{M}}_{0,n_v}) \prod_{e\in E_\Gamma} e^{-\vec{t}\cdot\vec{d}_e}\,. \qquad (3.1)$$

This is nothing else but the critical value of the action functional:

$$S(\phi) = -\frac{1}{2}\sum_{d,\langle ij\rangle} e^{t_{ij}d}\phi_{ij,d}\phi_{ji,d} + \sum_{k\geq 1}\frac{1}{k!}\chi(\overline{\mathcal{M}}_{0,k}) \sum_{i,\{(j_1,d_1)\dots(j_k,d_k)\}} \phi_{ij_1,d_1}\dots\phi_{ij_k,d_k}\,, \qquad (3.2)$$

where i,j label the vertices of the polygon (or the fixed points of the torus action on S), the "field" $\phi_{ij,d}$ is non-zero only iff i and j are connected by an edge, $d\in\mathbb{N}$. Introduce the following notations:

$$\begin{aligned} \phi_i &= \sum_{j,d}\phi_{ij,d} \\ \chi(\varphi) &= \sum_{k\geq 1}\chi(\overline{\mathcal{M}}_{0,k})\frac{\varphi^k}{k!} = \varphi + \frac{\varphi^2}{2} + \frac{\varphi^3}{6} + \cdots \end{aligned} \qquad (3.3)$$

Then the action (3.2) can be written much more simply as:

$$S = -\frac{1}{2}\sum_{d,\langle ij\rangle} e^{t_{ij}d}\phi_{ij,d}\phi_{ji,d} + \sum_i \chi(\phi_i)\,. \qquad (3.4)$$

Its critical points are the solutions to the following system of equations:

$$\begin{aligned} \phi_{ji,d} &= e^{-t_{ij}d}\xi_i \\ \xi_i &= \chi'(\phi_i) \\ \phi_i &= \sum_{j\neq i,\langle ij\rangle} \frac{1}{e^{t_{ji}}-1}\xi_j\,. \end{aligned} \qquad (3.5)$$

In [4] the following representation for $\chi(\varphi)$ can be found:

$$\begin{aligned}\chi(\varphi) &= \varphi + \mathrm{Crit}_{\xi}\left(\mathcal{S}(\xi) + \xi\varphi\right)\\ \mathcal{S}(\xi) &= \frac{1}{2}(1+\xi)^2\log(1+\xi) - \frac{\xi}{2} - \frac{5\xi^2}{4}\,.\end{aligned} \tag{3.6}$$

Hence we can replace the action (3.2) by the new action which depends on extra variables ξ_i without affecting its critical value:

$$S(\phi) \to -\frac{1}{2}\sum_{d,\langle ij\rangle} e^{t_{ij}d}\phi_{ij,d}\phi_{ji,d} + \sum_i (1+\xi_i)\,\phi_i + \mathcal{S}(\xi_i)\,. \tag{3.7}$$

Now, since this action depends quadratically on $\phi_{ij,d}$ they can be eliminated, leaving us with the action which depends (non-linearly) only on a finite number of variables:

$$\Theta(\xi) = \frac{1}{2}\sum_{\langle ij\rangle}\frac{(1+\xi_i)(1+\xi_j)}{e^{t_{ij}}-1} + \sum_i \mathcal{S}(\xi_i)\,. \tag{3.8}$$

Notice that this is in a way the opposite to what was done in [3] in a similar situation.

3.2 Example: the projective space

Consider $S = \mathbb{P}^n$. The polygon Δ is the simplex with $n+1$ vertices f_i, labelled by $i = 0,\ldots,n$. All t_{ij} are equal to each other. Introduce the notation: $z = e^{-t_{ij}}, Q = z/(1-z)$. Then the formal minimum of the action (3.8) is obtained by setting all ξ_i to be equal to each other:

$$\Theta(\xi) = Q\frac{n(n+1)}{2}(1+\xi)^2 + (n+1)\mathcal{S}(\xi)\,. \tag{3.9}$$

The critical point of (3.9) obeys:

$$nQ(1+\xi) + (1+\xi)\log(1+\xi) - 2\xi = 0\,. \tag{3.10}$$

This equation is to be compared to the $t = 0$ case in [5] (The equation (2.11) there yields the same equation for the critical point as (3.10). Moreover, the critical value agree with the equation (2.1) of [5]).

3.3 Maps with marked points

Suppose we are interested in counting the stable maps with k marked points on the worldsheet. The stability condition involves the marked points, so the answer will be slightly different. For example, a curve without marked points has no stable maps of degree zero, while the curve with three marked points has. It is easy to incorporate marked points, though. In the language of graphs we used so far the marked points can be thought as being the extra vertices, such that the edges which connect them to the rest of the graph carry no index d. This can be incorporated by the shift $\phi_i \to \phi_i + t$ in the action (3.4) together with the shift of the whole action by $\sum_i -t - t^2/2$ in order to remove the unstable configurations with only two or one marked point on the component which is sent to a point in the target space:

$$S = -\frac{1}{2} \sum_{d,\langle ij \rangle} e^{t_{ij} d} \phi_{ij,d} \phi_{ji,d} + \sum_i \left(\chi(\phi_i + t) - t - \frac{t^2}{2} \right). \tag{3.11}$$

The effective action becomes:

$$\Theta(\xi) = \frac{1}{2} \sum_{\langle ij \rangle} \frac{(1+\xi_i)(1+\xi_j)}{e^{t_{ij}} - 1} + \sum_i \left(\mathcal{S}(\xi_i) + t\xi_i - \frac{t^2}{2} \right). \tag{3.12}$$

The critical point now obeys:

$$(1+\xi_i)\log(1+\xi_i) - 2\xi_i + t = -\sum_{\langle ji \rangle} \frac{(1+\xi_j)}{e^{t_{ij}} - 1}. \tag{3.13}$$

In the example of $S = \mathbb{P}^n$ this reduces to

$$nQ(1+\xi) + (1+\xi)\log(1+\xi) - 2\xi + t = 0 \tag{3.14}$$

which again agrees with the equation (2.11) in [5]. It is also possible to express the critical value of the action as a polynomial in ξ_i:

$$\mathrm{Crit}\Theta = \frac{1}{4} \sum_i \left(1 - t^2 - (1 + t - \xi_i)^2\right). \tag{3.15}$$

This is the main result so far:

$$\sum_{k,\vec{d}} \chi(\overline{\mathcal{M}}_{0,k,\vec{d}}) \frac{t^k}{k!} e^{-\vec{t}\cdot\vec{d}} = \frac{1}{4} \sum_i \left(1 - t^2 - (1 + t - \xi_i)^2\right). \tag{3.16}$$

As a simple check, in the limit $t_{ij} \to \infty$ (degree zero maps) we get just the Euler characteristics of the moduli spaces of curves $\overline{\mathcal{M}}_{0,k}$ times the euler characteristics of the target S (again in agreement with (2.11) in [5]). In the sequel we shall use the identity:

$$\frac{d\mathrm{Crit}\Theta}{dt} = \sum_i (\xi_i - t). \tag{3.17}$$

4. Comparison to the results of mirror symmetry

We compare the results of the previous section to the predictions of the local mirror symmetry in the case: $S = \mathbb{P}^2$. The formula (3.15) implies:

$$\widetilde{\mathcal{F}}_0(z) := \sum_d \chi(\overline{\mathcal{M}}_{0,0,d}) z^d = \frac{3}{2} \left(\xi - \frac{1}{2}\xi^2 \right), \tag{4.1}$$

where ξ solves the equation:

$$f_z(\xi) = \frac{2z}{1-z}, \quad f_z(\xi) = 2\left(1 - \frac{z}{1-z}\right)\xi - (1+\xi)\log(1+\xi) = 0. \tag{4.2}$$

It is easy to expand the solution to (4.2) as a series in z: first write

$$\xi = \sum_{k=0}^{\infty} \left(\frac{2z}{1-z}\right)^k a_k(z)\,, \tag{4.3}$$

where

$$a_k(z) = \mathrm{Res}\,\xi \frac{df_z(\xi)}{f_z^{k+1}(\xi)} \tag{4.4}$$

and then re-expand in z. Explicitly we get:

$$\widetilde{\mathcal{F}}_0(z) = 3z + 9z^2 + 31z^3 + 121z^4 + \frac{2623}{5}z^5 + \frac{37111}{15}z^6 + \cdots \tag{4.5}$$

We may also consider the generating function for the Euler characteristics of the moduli spaces of curves with three marked points. In this case we get:

$$\widetilde{\mathcal{F}}_3(z) = 3 + 30z + 276z^2 + 2398z^3 + 20174z^4 + 166266z^5 + \frac{20263976}{15}z^6 + \cdots \tag{4.6}$$

On the other hand [6] got the expression (in our normalization):

$$\mathcal{F}(z) = 3z + 9z^2 + 30z^3 + 201z^4 + 1698z^5 + 17100z^6 + \cdots \tag{4.7}$$

(we added multiple coverings to the numbers we got from the Table 6 for $d = 0$ in [6]). We see that the numbers (4.7) are slightly bigger then those in (4.5) but are much smaller then those in (4.6). All this implies (together with the fact that the Euler characteristics comes out fractional for high degree maps, as the moduli spaces are really orbifolds) that there is a discrepancy and that the moduli space of wrapped $M2$-brane is different from merely the moduli space of holomorphic curves. Indeed, it was shown in [2] (using the results of [7]) that if $\mathcal{M}$ is the space of the collective coordinates of the soliton in M-theory which gives rise to a BPS particle in five dimensions then its contribution to the four dimensional beta function and correspondingly to the prepotential $\mathcal{F}$ is precisely $\chi(\mathcal{M})$. The resolution to this puzzle was suggested by C. Vafa [8]. The claim is that one should think of the wrapped $M2$-brane as of the wrapped $D2$-brane. The latter has extra moduli which come from the U(1) gauge field on the brane world-volume.

Here we present a few arguments in favor of this conjecture. The mysterious as it seems such an addition is natural if we try to perform the duality transformation which maps the gauge field to the extra scalar describing the motion of the brane in eleventh dimension more carefully. Since the duality transformation operates on the curvature of the gauge field: $F = \star d\phi_D$ it misses precisely the flat connections, i.e. the gauge fields A whose curvature vanishes: $F = dA = 0$.

Since the jacobian (the moduli space of flat connections) of the genus g curve is birationaly equivalent to the g'th symmetric power of the curve it is natural to try to parameterize the moduli space of pairs (C, L), where C is a holomorphic curve and L is a holomorphic line bundle on it by first choosing g points in S and then considering the space of curves which pass through these points. The latter space is isomorphic to $\mathbb{P}^d$ where d is the dimension of the moduli space of rational curve in a given homology class β: $d = \langle c_1(S), \beta\rangle - 1$.

One evidence in favor of this proposal comes from the study of the BPS states in the compactification of M-theory on K3 where one expects to see the spectrum of heterotic string compactified on T^3. In this case the virtual dimension of the moduli space of rational curves is negative. The symmetric product of K3's can be resolved to a smooth hyperkahler manifold, whose Euler characteristics is generated by the series:

$$\sum_g q^g \chi\left(\widetilde{\mathrm{Sym}_g(\mathrm{K3})}\right) = \frac{1}{\prod_n (1-q^n)^{24}}\,,$$

where one can recognize the right-moving oscillators of the heterotic string. This example shows that the symmetric product of the manifold has to be resolved (this resolution is needed for two purposes: first — in order to have a curve which passes through the coincident points one needs the tangent line to the curve at the point, and second - we mentioned above that the jacobian of the curve is not quite the symmetric product — they are related by a birational transformation). Unfortunately at this point we are unable to state precisely what is the resolution of the symmetric product which one has to consider. We present one more example (which was discovered by joint efforts with B. Kol), namely the curves in $S = \mathbb{P}^1 \times \mathbb{P}^1$. The curves have bi-degree (m, n) - the number of times they wrap the first and the second $\mathbb{P}^1$'s respectively. The dimension d in this case is equal to $2(m+n)$. The genus of generic curve in the (m, n) homology class is $g = (m-1)(n-1)$. The symmetric products of S (without resolution) have Euler characteristics which is generated by the series

$$\sum_g q^g \chi\left(\mathrm{Sym}_g(S)\right) = \frac{1}{(1-q)^4}$$

(i.e. 4 bosonic oscillators instead of the 4 right-moving chiral bosons as would have been the case for the complete resolution). It turns out that the partial resolution

$$\mathcal{X}(q) = \sum_g q^g \chi\left(\widehat{\mathrm{Sym}_g(S)}\right) = \frac{1}{(1-q)^4(1-q^2)} \tag{4.8}$$

does the job! More precisely, the series

$$\begin{aligned}\mathcal{F}_{(2,*)}(z,w) &= z^2 \sum_n w^n \left(2(n+2)\chi\left(\widehat{\mathrm{Sym}_{n-1}(S)}\right)\right) \\ &= 6w + 32w^2 + 110w^3 + 288w^4 + 644w^5 + 1280w^6 + 2340w^7 + \\ &\quad +4000w^8 + 6490w^9 + 10080w^{10} + 15106w^{11} + 21952w^{12} + \\ &\quad +31080w^{13} + 43008w^{14} + 58344w^{15} + 77760w^{16} + \cdots \end{aligned} \tag{4.9}$$

precisely coincides with the one in [9]. On the one hand this is very encouraging since we managed to count curves of arbitrary genus. On the other hand we were unable so far to extend this counting to, e.g. $(3,n)$ curves. This problem is currently under investigation [10].

5. Conclusions and speculations

5.1 Higher genus curves

To extend the techniques of the previous section to cover the higher genera we assume that the fixed point techniques can be applied there (the obstruction being the non-smoothness of the moduli spaces) (see [3] for discussion). Besides that we need to know $\chi(\overline{\mathcal{M}}_{g,n})$. The virtual Euler characteristics of the uncompactified moduli spaces $\mathcal{M}_{g,n}$ are known [11]: for $g > 1$

$$\chi_{g,n} = \chi(\mathcal{M}_{g,n}) = \frac{(2-2g)(2-2g-1)\cdots(2-2g-(n-1))}{n!} \frac{B_{2g}}{2g(2g-2)}, \tag{5.1}$$

where B_{2g} is the $2g$-th Bernoulli number. For genus one:

$$\chi_{1,n} = \frac{(-)^n}{12n}, \quad n > 0\,. \tag{5.2}$$

For genus zero:

$$\chi_{0,n} = \frac{(-)^{n-1}}{n(n-1)(n-2)}, \qquad n > 2\,. \tag{5.3}$$

This result is however insufficient for our purposes. We need to know the euler characteristics of the compactified moduli space with labelled punctures. The generating function of the numbers (5.1), (5.2) and (5.3) is:

$$\begin{aligned}\Sigma(\hbar,\xi) &:= \sum_{g\geq 0, n\geq 0, 2-2g-n<0} \chi_{g,n}\, \hbar^g \xi^n \\ &= \mathcal{S}(\xi) - \frac{\hbar}{12}\log(1+\xi) + \sum_{g>1} \frac{B_{2g}}{2g(2g-2)} \frac{\hbar^g}{(1+\xi)^{2g-2}}\,. \end{aligned} \tag{5.4}$$

Let $\mu = (1+\xi)/\sqrt{\hbar}$. The function $\Sigma(\hbar,\xi)$ has integral representation:

$$\Sigma(\hbar,\xi) = -\frac{3}{4} + 2\hbar^{1/2}\mu + \frac{\hbar}{4}\left(2\mu^2\log\mu - 3\mu^2 + \int_\epsilon^{\mu^{-1}} \frac{ds}{s\sinh^2(s/2)}\right), \tag{5.5}$$

where ϵ is a regulator and one should drop the singular part $1/3\log\epsilon - 2/\epsilon^2$ in the right hand side.

The conjecture I. The Euler characteristics of the compactified moduli spaces of curves $\chi(\overline{\mathcal{M}}_{g,n})$ are given by:

$$\Theta(t,\hbar) = \sum_{g,n} \frac{1}{n!}\chi(\overline{\mathcal{M}}_{g,n}) t^n \hbar^{g-1} = \log \int \frac{d\xi}{\sqrt{2\pi\hbar}} \exp \frac{1}{\hbar}\left(\Sigma(\hbar,\xi) + t\xi\right). \tag{5.6}$$

Remark. The conjecture is motivated by the following. The expansion of the integral (5.6) by Feynmann rules generates all possible connected graphs, whose vertices are labelled by the numbers g_v, s.t. $g_v \geq 0$, and which have the topological characteristics of the stable curve of genus g with n punctures. The genus g is obtained by summing the internal genera g_v and the number of loops in the graph. One should check that the graphs which correspond to unstable curves $2-2g-n \geq 0$ are not generated. In fact, the only possibility is to have a single loop without vertices, but it has weight zero which follows from the fact that $\mathcal{S}''(\xi) = -1$ for $\xi = 0$. In the limit $\hbar \to 0$ we can use saddle point approximation. Then (5.6) reduces to the statement proven in [4]. Notice that the action (5.5) is very much similar to the free energy of Penner model [11] although it differs from that in the unstable terms corresponding to the genera 0 and 1. Since we are to take the integral with this action we may say that the Euler characteristics of the compactified moduli spaces are given by the second quantized Penner model!
Now the extension for the stable maps is straightforward: we should sum over all graphs described in the section 2.2 and compute:

$$\Theta(\vec{t},\hbar) = \sum_{\Gamma,n,g,\vec{d}} \frac{\hbar^{g-1}}{\#\mathrm{Aut}(\Gamma)} \prod_{v\in V_\Gamma} \chi(\overline{\mathcal{M}}_{g_v,n_v}) \prod_{e\in E_\Gamma} e^{-\vec{t}\cdot\vec{d}_e}. \tag{5.7}$$

By Feynmann rules this is given by the (path) integral:

$$\Theta(\vec{t},\hbar) = \log \int \prod_{\langle ij\rangle,d} \mathcal{D}\left(\frac{\phi_{ij,d}}{\sqrt{2\pi\hbar}}\right) \exp \frac{1}{\hbar} S(\phi,\hbar,\vec{t}), \tag{5.8}$$

where the action is the modification of (3.2):

$$\begin{aligned} S(\phi,\hbar,\vec{t}) &= -\frac{1}{2}\sum_{d,\langle ij\rangle} \phi_{ij,d}\phi_{ji,d} + \\ &+ \sum_{g,k} \frac{\hbar^g}{k!}\chi(\overline{\mathcal{M}}_{g,k}) \sum_{i,\{(j_1,d_1)\dots(j_k,d_k)\}} \prod_{\ell=1}^{k} \phi_{ij_\ell,d_\ell} e^{-\frac{1}{2}t_{ij_\ell}d_\ell}. \end{aligned} \tag{5.9}$$

Thanks to the conjecture (5.6) we may rewrite the integral in (5.8) as the integral over the finite number of auxilliary variables ξ_i:

$$\int \prod_{\langle ij\rangle,d} \frac{\mathcal{D}\phi_{ij,d}\, e^{\frac{1+\xi_i}{\hbar}\phi_{ij,d} e^{-\frac{1}{2}t_{ij}d}}}{\sqrt{2\pi\hbar}} \prod_i \frac{\mathcal{D}\xi_i e^{\frac{\Sigma(\xi_i,\hbar)}{\hbar}}}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}\sum_{d,\langle ij\rangle}\phi_{ij,d}\phi_{ji,d}}. \tag{5.10}$$

Now the variables $\phi_{ij,d}$ enter the exponent quadratically and can be integrated out:

$$\int \prod_i \frac{\mathcal{D}\xi_i e^{\frac{\Sigma(\xi_i,\hbar)}{\hbar}}}{\sqrt{2\pi\hbar}} e^{\frac{1}{2\hbar}\sum_{\langle ij\rangle}\frac{(1+\xi_i)(1+\xi_j)}{e^{t_{ij}}-1}} . \tag{5.11}$$

Thus we arrive at:

Conjecture II. The euler characteristics of the moduli spaces of stable holomorphic curves in S are given by the logarithm of the integral (5.11).

The inclusion of the marked points is as straightforward as in the genus zero case:

Conjecture III.

$$\begin{aligned} \Theta_S(\vec{t}|t,\hbar) &= \sum_{n,g,\vec{d}} \chi(\overline{\mathcal{M}}_{g,n,\vec{d}}) \frac{t^n \hbar^{g-1}}{n!} e^{-\vec{t}\cdot\vec{d}} \qquad (5.12) \\ &= \log \int \prod_i \frac{\mathcal{D}\xi_i}{\sqrt{2\pi\hbar}} \exp \frac{1}{2\hbar}\Big(\sum_{\langle ij\rangle} \frac{(1+\xi_i)(1+\xi_j)}{e^{t_{ij}}-1} + \\ &\qquad + \sum_i \big(2\Sigma(\xi_i,\hbar) + 2t\xi_i - t^2\big)\Big). \end{aligned}$$

As a check, in the limit $t_{ij} \to \infty$ in which only degree zero maps survive we get:

$$\Theta_S(\vec{t}|t,\hbar) \to \chi(S)\Theta(t,\hbar)$$

as expected. It is amusing to note that the function $\Sigma(\xi,\hbar)$ also appears in [12] in a related context of $M2$-branes, the associated BPS particles and their quantum loops.

Remark. Although it is hard to compute the integral (5.12) it is still possible to get the genus one contributions without much extra work as long as genus zero is analysed. We hope to apply this observation elsewhere.

5.2 The last conlcusions and speculations

Let us summarize. We studied the genus zero curves which yield the BPS states in the compactifications of M -theory on a Calabi-Yau manifold. If the latter contains a surface S then the cuvres in S can be wrapped by $M2$-branes and become BPS particles in five dimensions. The states of the particles correspond to the cohomology of the moduli space of curves. We studied the Euler characteristics of the moduli spaces and found a representation for the generating function of these numbers in case where S is a toric variety. By making the comparison to the predictions of local mirror symmetry we observed the discrepancy which has to do with the fact that the moduli space contains the extra piece - the jacobians of the curves.[1] Apparently this question is an intriguing and worth further investigations. Our method of counting the curves can be applied to the higher genera as well. We presented our conjectures concerning this counting.

The last remark concerns the counting of curves in case where the target S is $\mathbb{P}^1$. In this case we may find a relation to the proposed string theory of two-dimensional QCD. Namely, it was suggested in [15] that the latter counts (in the chiral sector) the euelr characteristics of the moduli space of holomorphic maps of the string worldsheet into the target space where the large N Yang-Mills theory lives. Unfortunately it seems that the compactification of the moduli space of maps which is used to reproduce the QCD partition function differs from that of stable maps.

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